Echelon Forms

Sect 1.2

$$A = \begin{pmatrix} 3 & 0 & -2 & -5 \\ 0 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 9 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 6 & 4 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & -3 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 9 & 4 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1.2 Row Reduction and Echelon Form.

Previous to the definition:

Nonzero row or Nonzero Column:

It is a row or column that contains at least one nonzero number or entry.

Leading Entry of a Row:

It is the leftmost nonzero entry in a nonzero row.

Some important forms of matrices

Property 1:- All nonzero rows are above any row of all zeros.

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 3 & 4 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 5 & 3 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Property 2:- Each leading entry of a row is in a column to the right of the leading entry of the row above it.

$$\begin{pmatrix} 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & -5 & 4 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 3 & -5 & 4 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A Consenquence of properties 1 and 2 is

Property 3.- All entries in a column below a leading entry are zeros.

Definition.

A rectangular matrix is in echelon form or row echelon form if it satisfies properties 1 and 2.

Two more important properties:

Property 4.- The leading entry in each nonzero row is 1.

Property 5.- Each leading 1 is the only nonzero entry in its column.

The previous matrices are in row echelon form but they do not satisfy properties 4 and 5.
In fact,

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 5 & 3 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 3 & -5 & 4 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

However, by performing some more row operations they can be transformed into "row-equivalent" matrices that satisfy properties 4 and 5

In fact,

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 5 & 3 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{\frac{\text{row } 2}{5}}{\sim} \begin{pmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 1 & \frac{3}{5} & \frac{4}{5} & \frac{-1}{5} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{(-2)\text{row } 2 + \text{row } 1}{\sim} \boxed{\begin{pmatrix} 1 & 0 & -\frac{6}{5} & -\frac{8}{5} & \frac{22}{5} \\ 0 & 1 & \frac{3}{5} & \frac{4}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}$$

Also,

$$B = \begin{pmatrix} 0 & 1 & 3 & -5 & 4 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{\frac{\text{row } 2}{2},\ \frac{\text{row } 3}{4}}{\sim} \begin{pmatrix} 0 & 1 & 3 & -5 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim$$

$$\overset{(-3)\text{row } 2 + \text{row } 1}{\sim} \begin{pmatrix} 0 & 1 & 0 & -5 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{(5)\text{row } 3 + \text{row } 1}{\sim} \boxed{\begin{pmatrix} 0 & 1 & 0 & 0 & \frac{41}{4} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}$$

<u>Definition</u> A rectangular matrix in echelon form is in <u>reduced row echelon form</u> or <u>reduced echelon form</u> if it satisfies properties 4 and 5.

Section 1.2

Problem #3

Leading entry 1^{st} row

first non-zero entry in column 1

$$A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & -3 & -12 \end{bmatrix} \sim$$

(-2) row1 + row 2
(-3) row1 + row 3

$\left(-\frac{3}{2}\right)$ row 2 + row 3

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\left(-\frac{1}{2}\right)$ row 2

(-4) row 2 + row 1

Pivot positions

Pivot positions

Pivot Positions

$$A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

Pivot columns. 1 and 3.

Pivot positions

Echelon form of A

Reduced Row echelon form of A

Definition

Two matrices are called row equivalent if there is a Sequence of elementary row operations that transforms one matrix into the other.

Theorem 1 Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only One reduced echelon matrix.

Observation: For any **echelon** form matrix of a given matrix the leading entries are always in the same position.

Definition

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

A pivot Column is a column of A that contains a pivot position.

From the previous observation and definition, we can also define a pivot position as a location in the original matrix A that corresponds to a leading entry in any echelon form of A.

If the matrix A were the augmented matrix of a linear system then, this linear system would be

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 8 \\ 2x_1 + 4x_2 + 6x_3 &= 8 \\ 3x_1 + 6x_2 + 9x_3 &= 12 \end{aligned} \qquad A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

After the row reduction

$$\tilde{A} = \begin{bmatrix} 1 & 2 & 0 & -8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 + 2x_2 &= -8 \\ x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

$x_1 = -8 - 2x_2$ free Variable

$x_3 = 4$

Basic Variables

They correspond to pivot positions.

General Solution

$$\begin{cases} x_1 = -8 - 2x_2 \\ x_2 \text{ free} \\ x_3 = 4 \end{cases}$$

Infinitely many solutions. they depend on the value of x_2.

If $x_2 = 1 \Rightarrow x_1 = -10, \; x_3 = 4.$

If $x_2 = -1 \Rightarrow x_1 = -6, \; x_3 = 4$

Two solutions are

$$\begin{cases} x_1 = -10 \\ x_2 = 1 \\ x_3 = 4 \end{cases}, \quad \begin{cases} x_1 = -6 \\ x_2 = -1 \\ x_3 = 4 \end{cases}$$

Definition

The variables x_1 and x_3 in problem # 3 (Sect. 1.2) that correspond to pivot positions are called BASIC VARIABLES. Those variables which do not correspond to pivot positions are called free variables. In problem # 3 x_2 is a free variable.

Definition

A rectangular matrix in echelon form is in Reduced Echelon form or Reduced Row Echelon form if it satisfies properties 4 and 5.

Solutions of Linear Systems from Augmented matrix in row echelon form.

$$\begin{pmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \begin{cases} x_3 = 1 \\ x_2 = 2 - 3x_3 = 2 - 3(1) = -1 \\ x_1 = -5 + 2x_3 = -5 + 2 = -3 \end{cases}$$

There are no free variables

"Unique Solution"

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 5 & -4 \end{pmatrix} \qquad \begin{cases} x_3 = -4 - 5x_4 \\ x_2 = 2 - 3x_4 \\ x_1 = -1 - 2x_4 \end{cases}$$

x_4 is called a free Variable.

"Infinitely many solutions"

$$\begin{pmatrix} 1 & 2 & 0 & -11 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$0 = 0$

$0 = 1$ Contradiction!

$x_3 = 5$

$x_1 = -11 - 2x_2$

"No Solutions"

THEOREM 2 Existence and Uniqueness Theorem

1. *A linear system is consistent if and only if the rightmost column of the augmented matrix is* **not** *a pivot column, that is, if and only if an echelon form of* the *augmented matrix has no row of the form* [0 . . . 0 b], *with* b ≠ 0.

2. *If a linear system is consistent, then the solution set contains either*

   (i) *a unique solution, when there are no free variables, or*

   (ii) *infinitely many solutions, when there is at least one free variable.*